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Herbert F. Wang: Theory of Linear Poroelasticity with Applications to Geomechanics and Hydrogeology

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Introduction

1.0 CHAPTER OVERVIEW

The purpose of this book is to quantify, in the simplest linear approximation, how fluid extraction or injection produces stress changes in isotropic, fluid-saturated rock formations; and conversely, how loads applied to an aquifer by engineered structures, atmospheric pressure, earth and ocean tides, tectonic activity, or reservoir lake level changes produce water-level changes in wells. This coupling between changes in stress and changes in fluid pressure forms the subject of poroelasticity. Literal clues to poroelastic phenomena appear in metaphors such as “stressing an aquifer” in reference to pumping a well or “strain meter” in reference to a water well on a fault.

The term *poroelasticity* was first coined by J. Geertsma as a footnote in his 1966 paper entitled “Problems of rock mechanics in petroleum production engineering.” Geertsma’s footnote refers specifically to “Biot’s work on the theory of the elasticity and viscoelasticity of fluid-saturated porous solids” as “typical of the macroscopic stress-strain relations to be encountered.” Geertsma explicitly pointed out that “the mathematical description of the macroscopic theory of poroelasticity is similar to that used in the theory of thermoelasticity.” This chapter provides a historical perspective on how the disciplines of geomechanics, hydrogeology, and petroleum engineering contributed to our present understanding of coupled fluid and mechanical behavior.

This chapter previews the constitutive equations of poroelasticity for the case of an isotropic applied stress field. It examines the physical significance of material coefficients such as drained compressibility, poroelastic expansion coefficient, specific storage coefficient, and Skempton’s coefficient. The chapter concludes with a description of the analogy between poroelasticity and thermoelasticity.

1.1 HISTORICAL EXAMPLES

Poroelastic behavior can explain an initially unexpected connection between a causal event and its subsequent effect. What follows is a variety of historical

examples of poroelastic phenomena:

- *Water Levels in Wells Correlate with Ocean Tides.* Pliny the Elder stated in A.D. 77 that near the temple of Hercules in Cadiz “there is a closed source similar to a well which occasionally rises and falls with the ocean, but at other times does the opposite” (Melchior, 1983, p. 2).
- *Water Levels Change in Well as Trains Pass.* F. H. King (1892) of the University of Wisconsin reported that water levels in a well near the train station at Whitewater, Wisconsin, went up as a train approached and went down as a train left the station. The water level fluctuation was greater for a heavy freight train than for a lighter and faster passenger train.
- *Water Levels in Boardwalk Wells Fluctuate with Ocean Tides.* In 1902 the United States Geological Survey (USGS) reported that water-level oscillations in wells in Atlantic City, New Jersey, were synchronous with ocean depths, because the weight of sea water at high tide compressed the underlying rock, thereby forcing pore water up the wells (Meinzer, 1928).
- *Texas Claims Oil Fields That Sink into the Sea.* The state of Texas claimed title to the part of the Goose Creek oil field near Galveston, Texas, that had become covered with water from Galveston Bay following oil production. The state used the argument that the submerged land belonged to the state. The counterclaim by landowners was based on a geologic study by Pratt and Johnson (1926, p. 582) showing that the subsidence could be attributed to the extraction of 100 million barrels of water, oil, and sand from the reservoir. The courts ruled against the state claim because the submergence was due to human action and not natural causes.
- *Water Levels Rise in Wells Near a Pumping Well.* Observers in the small fishing village of Noordbergum in northern Friesland (Netherlands) witnessed a curious rise in water levels when large pumps were turned on in nearby wells. The behavior lasted a few hours before the water levels dropped. When the large pumps were shut off, the reverse situation occurred—namely, water levels dropped farther before they recovered. Verruijt (1969) concluded that the reverse well fluctuations occurred because pumping instantly compressed the aquifer to force water up the well.
- *Lake Mead Triggers Earthquakes.* The completion in 1935 of Hoover Dam along the Colorado River created 100-meter-deep Lake Mead behind it. Shortly after completion, small earthquakes beneath the lake occurred as a result of a combination of factors (Roeloffs, 1988). One factor was that the additional weight of the dam and water was stressing faults to the failure point; another was that water from

the lake seeped into the fault, thereby reducing their resistance to slipping.

1.2 BASIC CONCEPTS

Two basic phenomena underlie poroelastic behavior:

- *Solid-to-fluid coupling* occurs when a change in applied stress produces a change in fluid pressure or fluid mass.
- *Fluid-to-solid coupling* occurs when a change in fluid pressure or fluid mass produces a change in the volume of the porous material.

As used here, the word *solid* refers to the skeletal framework of bulk, porous material. The terms *fluid pressure* and *pore pressure* are also used interchangeably. The solid-to-fluid and fluid-to-solid couplings are assumed to occur instantaneously in the quasistatic approximation in which elastic wave propagation is ignored. The simplest mathematical description of the two basic forms of coupling between solid and fluid is a set of linear constitutive equations. The equations relate strain and fluid-mass changes to stress and fluid-pressure changes. The poroelastic constitutive equations are generalizations of linear elasticity whereby the fluid pressure field is incorporated in a fashion entirely analogous to the manner in which the temperature field is incorporated in thermoelasticity (cf. Section 1.9). An increase of fluid pressure causes the medium to expand just as an increase of temperature causes it to expand.

The magnitude of the solid-to-fluid coupling depends on the compressibility of the framework of the porous material, the compressibility of the pores, the compressibility of the solid grains, the compressibility of the pore fluid, and the porosity. Negligible solid-to-fluid coupling occurs for a highly compressible fluid such as air. An example of solid-to-fluid coupling is the response of water levels in a well to the passage of nearby trains (Fig. 1.1).

Changes of fluid mass or fluid pressure in a porous material produce strains in the bulk, porous solid. A uniform change in fluid pressure throughout a porous body subjected to no external stresses or constraints (*free strain case*) produces a uniform strain and no poroelastic stresses. In general, consideration of boundary constraints means that the strain field is different from the free strain case, and poroelastic stresses exist within the body. An example of fluid-to-solid coupling is subsidence due to large amounts of fluid extraction from an aquifer or hydrocarbon reservoir (Fig. 1.2).

A nonuniform pore pressure distribution leads to time-dependent fluid flow according to Darcy's (1856) law. The time dependence of pore pressure pro-

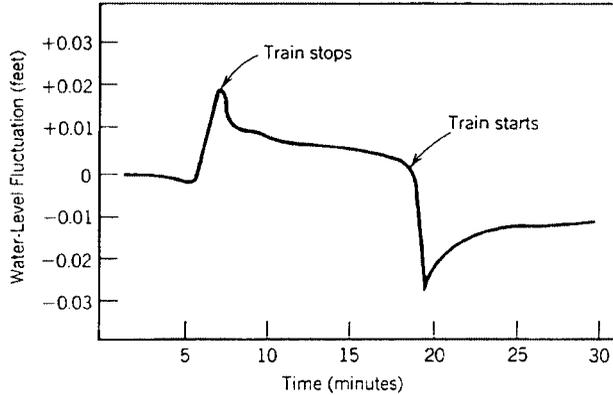


Figure 1.1: Water-level fluctuations due to a passing train. An approaching train compresses the aquifer, which quickly raises the pore pressure in the affected region. Fluid then flows away from the high-pressure region. As the train departs, the aquifer expands, thereby quickly reducing the pore pressure in the affected region. Fluid again flows in response to the pressure differences, but this time it builds up the pore pressure. The approximately equal and opposite behaviors demonstrate that the aquifer is elastic (Domenico and Schwartz, 1998, p. 65; Jacob, 1940).

duces time dependence of poroelastic stresses and strains, which in turn couple back to the pore pressure field. Quantifying these basic poroelastic concepts for application to time-dependent, coupled deformation and fluid-flow problems in hydrogeology, geomechanics, and petroleum engineering is the subject of this book. If only fluid-to-solid coupling were important, the problem would be mathematically simpler because the fluid-flow problem can be solved independently of the stress field. The stress field (and hence strain and displacement fields) could then be calculated as functions of position and time once the pore pressure field has been determined as a function of position and time. This one-way coupling is called the *uncoupled* problem. However, when the time-dependent changes in stress feed back significantly to the pore pressure, the two-way coupling is important, and the problem is called *coupled*. In the mathematically analogous subject of thermoelasticity, significant heating of material from stress changes does not occur for most materials, and hence most thermoelastic analyses ignore this direction of coupling, and are uncoupled. However, applied stress changes in fluid-saturated porous materials typically produce significant changes in pore pressure, and this direction of coupling is significant.

1.3 BRIEF HISTORY

Important concepts of poroelasticity developed somewhat independently in geomechanics, petroleum engineering, and hydrogeology. One theme of this book is to highlight the unified description that poroelastic theory provides all three disciplines.

The scientific history of poroelastic concepts spans approximately one hundred fifty years. The early period, from the publication of Darcy's law¹ in 1856 to 1900, included observations of well behavior in response to various loading phenomena such as tides. Increased exploitation of groundwater and hydrocarbon resources in the 1900–1930 period was the impetus for improved scientific and engineering understanding of the principles governing their occurrence and flow. Also, increased civil construction during this period became the impetus for improved understanding of the behavior of soil as a foundation material. The canonical geomechanics problem was soil consolidation, and the canonical hydrogeology problem was elastic storage in a confined aquifer. From 1930 to 1960, significant progress was made in each of the three disciplines in developing fundamental concepts, formulating or extending constitutive laws and governing equations, and obtaining analytical solutions. After 1960, more complex analytical solutions were obtained for problems in land subsidence and earthquake mechanics. Numerical solutions increased in importance as the digital computer enabled more realistic simulations of geological situations and of nonlinear and time-dependent constitutive properties.

1.4 GEOMECHANICS

Karl Terzaghi (1883–1963) sought to understand the behavior of soil as a foundation material by performing controlled laboratory experiments. The work, which led to his consolidation theory,² was conducted between 1916 and 1925, when he was assigned by the Austrian Department of Foreign Affairs in Vienna to lecture at Robert University in Istanbul. He published two influential textbooks, *Erdbaumechanik auf bodenphysikalischer Grundlage* in 1925 and *Theoretical Soil Mechanics* in 1943. The key experiment from which he developed the governing Eqn. 1.1 is shown schematically in Figure 1.3. A fully saturated soil sample is confined laterally in a cylinder of

¹ An entertaining and illuminating biographical sketch of Darcy is given by Freeze (1994).

² A bitter scientific dispute developed between Terzaghi and Fillunger initially over the theory of uplift in dams and subsequently over the theory of consolidation. Fillunger committed suicide in 1937 after learning that a committee of experts would support Terzaghi's theory (Skempton, 1960; de Boer and Ehlers, 1990; de Boer, 1996; de Boer et al., 1996).

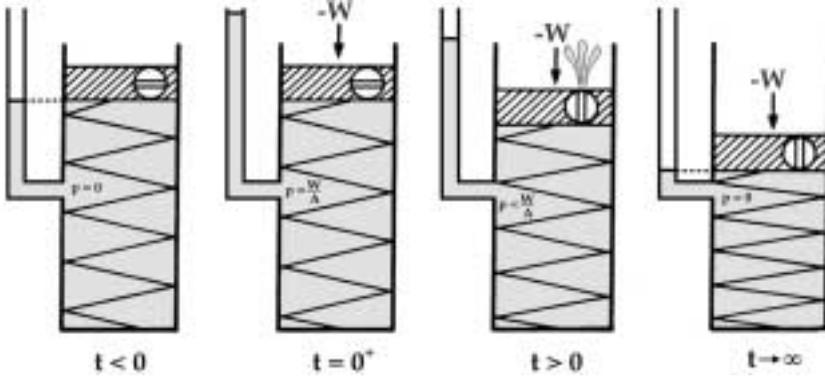


Figure 1.3: Schematic of uniaxially constrained soil consolidation (after Craig, 1997, p. 86). A compressive load $-W$ is applied suddenly at time $t = 0$ to a uniaxially confined sample of cross-sectional area A . The excess fluid pressure jumps to its undrained value W/A to support the load. Stress is transferred partially to the solid skeleton of the porous material (represented by the spring) until excess fluid pressure is again zero for long times and the load is carried entirely by the solid framework.

cross-sectional area A . An axial load $-W$ is applied suddenly at $t = 0$ and then held constant. (Tensile stresses are taken to be positive.) The water pressure throughout the sample jumps up by the amount $p = W/A$ at $t = 0^+$. A profile of excess pressure develops within the sample as water flows out the top drain, which is maintained at atmospheric pressure. Terzaghi derived the consolidation equation for this experiment to be the diffusion equation for excess (greater than hydrostatic) water pressure p ,

$$\frac{\partial p}{\partial t} = c \frac{\partial^2 p}{\partial z^2} \quad (1.1)$$

where c is a diffusivity that is known as the consolidation coefficient, t is time, and z is distance along the soil column.

As will be demonstrated in Section 6.3, Eqn. 1.1 is independent of stress, because the theory of poroelasticity leads to the special result that the pore pressure field and applied stress field are uncoupled for the boundary conditions in Terzaghi's experiment. The time evolution of the pressure profile is exactly the same as the analogous thermal conduction problem of a sudden step change (Carslaw and Jaeger, 1959, pp. 96–97), which was noted by Terzaghi.

Terzaghi is generally recognized for elucidating the important concept of effective stress, which for soils is well approximated to be the difference between the applied stress and pore pressure, because the grains are incom-

pressible relative to the bulk soil. Initially, the axial load is borne entirely by the fluid, but it is shifted to the skeletal frame as the excess pore pressure dissipates. A discussion of the effective stress concept is presented in Section 2.8.

1.5 HYDROGEOLOGY

Whereas Terzaghi sought general principles in the laboratory, developments regarding the poroelastic behavior of aquifers were based primarily on field observations. The first published record of water wells responding to passing trains was made by F. H. King (1892) in Wisconsin. He noted that a heavy freight train produced a greater rise in water level than did a lighter and faster passenger train, and that a locomotive alone did not produce a noticeable effect. O. E. Meinzer, in his 1928 paper “Compressibility and elasticity of artesian aquifers,” sought to resolve the puzzle of the source of the large amounts of irrigation water pumped from the Dakota sandstone. He reasoned that recharge was insufficient to produce 3000 gallons per minute for 38 years within three townships, and that the compressibility of water alone was likewise insufficient. He concluded that elastic aquifer compression occurred as a result of the decline in fluid pressure, and that the reduction in pore volume was the principal source of water released from storage. He also recognized that aquifers were elastic because well levels recovered after they were shut down. In support of his hypothesis, Meinzer cited King’s work on train-induced fluctuations in water levels, the in-phase response of water well levels to ocean tides in Atlantic City, New Jersey (Fig. 1.4), and the subsidence of the Goose Creek oil field. Meinzer also explicitly referenced Terzaghi’s effective stress principle to equate pressure decline to an effective stress increase. Meinzer’s insight was that he recognized that aquifers were compressible and that the laboratory-derived principle of effective stress could be applied to aquifers. No equations were presented in Meinzer’s 1928 paper, although he incorporated calculations for the relative amounts of water released from aquifer compression versus water expansion.

The next development in hydrogeology was Theis’s nonequilibrium or transient flow solution for drawdown of a well pumped at a constant rate. Theis first conceptualized the problem in terms of heat conduction and then sought advice from a former college classmate, Clarence Lubin, a mathematics professor at the University of Cincinnati:³

The flow of ground water has many analogies to the flow of heat by conduction. We have exact analogies in ground water theory for thermal

³ This passage is in a letter from Theis to Clarence Lubin dated December 19, 1934 (White and Clebsch, 1994, p. 51).

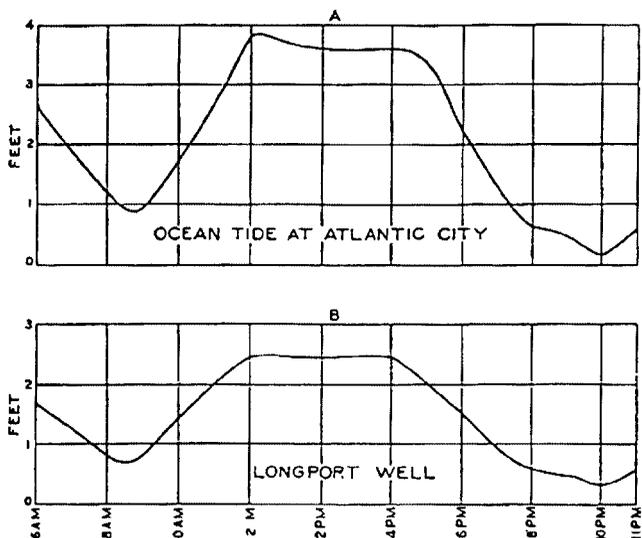


Figure 1.4: Ocean tides at Atlantic City. The in-phase response of well levels to ocean tides on January 22, 1926, in an 800-foot well near Longport, New Jersey, is evidence of solid-to-fluid coupling. A lag time would be evident if the well levels were responding to fluid flow (Meinzer, 1928, p. 274).

gradient, thermal conductivity, and specific heat. I think a close approach to the solution of some of our problems is probably already worked out in the theory of heat conduction. Is this problem in radial flow worked out? Given a plate of given constant thickness and with constant thermal characteristics at a uniform initial temperature to compute the temperatures throughout the plate at any time after the introduction of a sink kept at 0 temperature? And a more valuable one from our standpoint: Given the same plate under the same conditions to compute the temperatures after the introduction of a sink into which heat flows at a uniform rate? I forgot to say that the plate may be considered to have infinite areal extent.

Lubin provided Theis the solution from the 1921 edition of Carslaw's treatise on heat conduction. He declined Theis's offer to be a coauthor with the words "from the standpoint of mathematics the work is not of fundamental importance" (Freeze, 1985, p. 442). In the resulting paper, Theis (1935) stated the analogy between groundwater storage and specific heat:

In heat-conduction a specific amount of heat is lost concomitantly and instantaneously with fall in temperature. It appears probable,

analogously, that in elastic artesian⁴ aquifers a specific amount of water is discharged instantaneously from storage as the pressure falls.

Thus, Theis recognized that confined aquifers possess a property analogous to heat capacity, which he called the *coefficient of storage* in a subsequent paper (Theis, 1938). Jacob (1940) derived the transient flow equation for a horizontal aquifer “from scratch,” as he put it, rather than appeal to the heat flow analogy. Jacob translated Theis’s verbal definition of coefficient of storage \mathcal{S} into a mathematical definition:

$$\mathcal{S} = \frac{1}{\rho_f A} \frac{\delta M_f}{\delta h} \quad (1.2)$$

where ρ_f is the density of water, A is the horizontal cross-sectional area of a vertical column of aquifer, δM_f is the change in the mass of water in a column of area A , and δh is the change in head. Eqn. 1.2 is a macroscopic definition, which gives no insight into the physical mechanisms behind storage capacity. Jacob, therefore, considered the micromechanics⁵ of the release of water from aquifer compression and fluid expansion. With the assumption that the individual mineral grains are incompressible, Jacob derived the expression

$$\mathcal{S} = \rho_f g b \left(\frac{\phi}{K_f} + \frac{1}{K_v} \right) \quad (1.3)$$

where g is the acceleration of gravity, ϕ is the porosity, b is the aquifer thickness, K_f is the bulk modulus of water, and K_v is a modulus of compression of the aquifer. Jacob made no explicit specification of the particular modulus of compression represented by K_v ; he did not restrict it to be a vertical modulus.

Jacob (1950) later arrived at the three-dimensional governing equation, which is the usual starting point for transient fluid-flow analyses for confined aquifers,

$$\frac{\partial h}{\partial t} = \frac{\rho_f g k}{\mu S_s} \nabla^2 h \quad (1.4)$$

where k is the permeability, μ is the fluid viscosity, and S_s is the standard hydrogeological specific storage (Hantush, 1960). The fluid-flow equation obtained from poroelastic theory contains an additional term associated with the time dependence of strain of the elementary volume under consideration.

⁴The word *confined* is today generally used in place of *artesian*.

⁵Micromechanics is the science of mechanics applied to granular materials at the grain and pore scale.

This additional term was a decade-long issue in hydrogeology in the 1960s until 1969, when Verruijt showed that the most general (linear) description of aquifer behavior is obtained from Biot's theory.

1.6 PETROLEUM ENGINEERING

Muskat wrote *Flow of Homogeneous Fluids through Porous Media* over a period of about 5 years in the 1930s with the encouragement of his employer, Gulf Oil in Pittsburgh. His book can be viewed as a treatise on applied potential theory. Muskat systematically developed solutions to Laplace's and Poisson's equations and the diffusion equation for boundary and initial conditions relevant to problems of groundwater flow in aquifers and fluid movement in hydrocarbon reservoirs. Although he was affiliated with an oil company, his book provides a balanced coverage of literature in both hydrogeology and petroleum engineering. He included many problems he had solved personally. Muskat considered storage effects to result primarily from fluid expansion, since rocks were thought to be incompressible at reservoir depths. Fluid expansion, however, was inadequate to account for the 500 million barrels of oil extracted with a pressure decline of 375 pounds per square inch from the East Texas Oil Field in the late 1920s and early 1930s. Muskat attributed the high production to be due to drive from small pockets of highly compressible gas or to "water drive" from the peripheries of the reservoir. Jacob (1940) suggested alternatively that the high production was due to the compressibility of the Woodbine sand and clays.

Poroelastic theory was used in petroleum engineering primarily to understand subsidence, estimate hydrocarbon volumes, and predict stresses around boreholes. The subsidence of the Goose Creek oil field described by Pratt and Johnson (1926) was the first conceptual realization of the coupling between large volumes of fluid extraction and large-scale mechanical deformation. Because the problem of large groundwater withdrawals from aquifers is identical, Geertsma, who was affiliated with Shell in the Netherlands, referred in his 1966 paper to subsidence in Mexico City and the Houston-Galveston region in Texas. The groundwater subsidence literature in the 1970s (e.g., Gambolati and Freeze, 1973; Gambolati et al., 1974) used numerical techniques more heavily than the analytical techniques introduced by Geertsma and used most recently by Segall (Segall, 1992; Segall et al., 1994). Segall's work connects hydrocarbon extraction with induced seismicity.

Poroelasticity research in the 1940s and 1950s was oriented primarily toward rock mechanics. Hughes and Cooke (1953) made laboratory measurements of pore compressibilities to correct for available pore space at reservoir depths. Laboratory measurements of poroelastic parameters continue (e.g., Laurent et al., 1993; Hart and Wang, 1995). Of particular importance in

petroleum engineering and geophysics is the use of hydraulic fracturing as a technique for measuring earth stresses (Hubbert and Willis, 1957; Haimson and Fairhurst, 1969; Detournay and Cheng, 1993).

1.7 BIOT'S THEORY

Terzaghi's mathematical treatment was based on his *one-dimensional* laboratory experiments. The general *three-dimensional* theory of poroelasticity⁶ was formulated by Biot in 1941 when he was a professor of mechanics at Columbia University. It appeared in the *Journal of Applied Physics* with the title "General theory of three-dimensional consolidation" (Biot, 1941a). It contained the constitutive equations that are the central subject of this book. Biot subsequently extended the theory to describe wave propagation in fluid-filled porous media (Biot, 1956a, 1956b).

In his 1941 paper Biot introduced a quantity he called the *variation in water content*, which he defined as "the increment of water volume per unit volume of soil" (Biot, 1941a). The increment of water content is the volume of the water *exchanged* by flow into or out of the control volume. In other words, Biot's increment of water content is the volume of water added to storage as used in the earlier hydrogeologic work of Theis or Jacob, although Biot's work appears to be independent of theirs. Biot's variable for a generic fluid is called the *increment of fluid content* ζ . No symbol has been used generally in the hydrogeologic literature for this quantity. Biot's (1941a) description of a hypothetical experiment in which a thin tube is used to extract water from a soil sample evokes the image of withdrawing water from a well:

... consider a sample of soil enclosed in a thin rubber bag so that the stress applied to the soil be zero. Let us drain the water from this soil through a thin tube passing through the walls of the bag. If a negative pressure $-p_o$ is applied to the tube a certain amount of water will be sucked out. This amount is given by $\zeta = -p_o/R$.

A negative value of ζ indicates the removal of water. The proportionality constant $1/R$ is called the *specific storage coefficient at constant stress*, because it is the ratio of the change in increment of fluid content ζ to the change in pore pressure p for a stress-free sample. The specific storage coefficient at constant stress is also called the *unconstrained specific storage coefficient*, or the *three-dimensional specific storage coefficient*.

Four poroelastic moduli, rather than the usual two moduli in standard elasticity, are necessary to relate the strains and increment of fluid content to

⁶ Rendulic (1936) formulated a three-dimensional theory by substituting the Laplacian for the second spatial derivative in Eqn. 1.1. In three dimensions, this ad hoc generalization ignores an additional stress-coupling term.

Maurice Anthony Biot (1905–1985). Biot graduated with a bachelor's degree in philosophy (1927); degrees in mining engineering (1929) and electrical engineering (1930), and a D.Sc. (1931), all from Louvain University in Belgium, followed by a Ph.D. in aeronautical science (1932) from the California Institute of Technology under the supervision of Theodore von Kármán (Williams, 1983; Tolstoy, 1986; Cheng et al., 1998). Biot held academic positions at Harvard (1934–1935), Louvain (1935–1937), Columbia (1937–1946), and Brown (1946–1952), after which he became an independent researcher and consultant. Shell Development and Cornell Aeronautical Laboratory are two affiliations Biot used in his publications.

Biot worked on both sides of the poroelasticity-thermoelasticity coin, often citing the isomorphism between the two theories. Twenty-one of his papers were collected in a volume edited by Ivan Tolstoy and published by the Acoustical Society of America (Tolstoy, 1992). All but 2 of Biot's 175 papers have been collected into a CD-ROM by Thimus et al. (1998). Biot's work was in the nineteenth-century tradition of natural philosophy and mathematical physics. These roots are acknowledged in the preface of von Kármán and Biot's (1940) textbook *Mathematical Methods in Engineering* in their quote from Lord Kelvin and Peter Guthrie Tait's *Treatise on Natural Philosophy*, Part II: "Neither seeking nor avoiding mathematical exertitions we enter into problems solely with a view to possible usefulness for physical science." Equally apt historical quotations open each chapter of the text.

stresses and pore pressure. The constant $1/R$ is one of the two new constants. The other new constant $1/H$ is a *poroelastic expansion coefficient*, which relates the volumetric strain to pore pressure changes for conditions of constant stress. To the author's knowledge, the terminology *poroelastic expansion coefficient* appears to be new here, as this coefficient is often called *compressibility* (cf. Section 3.1).

Biot showed that Terzaghi's one-dimensional consolidation problem is a special case of his theory. McNamee and Gibson (1960a, 1960b) used Biot's theory to obtain analytical solutions for consolidation of a half space due to a strip or circular load. Other important applications of Biot's theory were to subsidence (Geertsma, 1966) and hydraulic fracturing (Haimson and Fairhurst, 1969). In 1969 in a paper entitled "Elastic storage of aquifers," Verruijt recognized the general applicability of Biot's theory to aquifer behavior. Verruijt opened his paper with the statement that parallel developments occurred in soil mechanics and groundwater hydrology.

Rice and Cleary's 1976 reformulation of Biot's linear poroelastic constitutive equations has been adopted widely for geophysical problems (Wang,

Specific Storage Coefficient of a Balloon. This simple experiment to measure the specific storage coefficient $1/R$ of a water-filled balloon (Fig. 1.7) provides observational meaning to the variables increment of fluid content and storage. A balloon is stretched over the end of a burette clamped to a meter stick and attached to a tall ring stand. The balloon and burette are filled with a volume V_f of water to an arbitrary height h_1 on the meter stick and volume V_1 on the burette. The volume of water in the balloon is $V_b = V_f - V_1$. Adding a measured volume of water ΔV_f from a beaker raises the water height to h_2 and increases the volume to V_2 in the burette. If the compressibility of water and the burette are assumed to be small, the volume of water ΔV_f is divided between the balloon and the burette: $\Delta V_f = \Delta V_b + (V_2 - V_1)$, where ΔV_b is the additional volume of water in the balloon. Because the increment of fluid content is the ratio $\Delta V_b/V_b$, $\zeta = [\Delta V_f - (V_2 - V_1)]/V_b$. The increase in pressure is $p = \rho_f g(h_2 - h_1)$. The specific storage coefficient $1/R$ is the ratio of ζ to p .

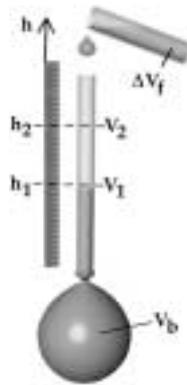


Figure 1.5: Apparatus to measure the specific storage coefficient of a water-filled balloon.

1993). Rice and Cleary chose constitutive parameters that emphasized the drained (constant pore pressure) and undrained (no flow) limits of long- and short-time behavior, respectively. Their perspective was from the field of applied mechanics, similar to that of Biot's, but they apparently were unaware of his much earlier contributions when they initially approached the problem. Rice and Cleary defined *fluid mass content* m_f to be the fluid mass per unit reference volume (see also Biot, 1973, p. 4930). The *change* in fluid mass content $\delta m_f = m_f - m_{f_0}$, where m_{f_0} is the fluid mass content in the

reference state, is related to increment of fluid content ζ by

$$\zeta = \frac{\delta m_f}{\rho_{f_o}} \quad (1.5)$$

where ρ_{f_o} is the fluid density in the reference state. Fluid mass content is a state property, whereas the increment of fluid content must be viewed in the hydrogeologic sense of volume of fluid *transported* into or out of storage. Jacob (1940) defined storage in terms of fluid mass (Eqn. 1.2), but an advantage to using ζ as a primary variable is that it is dimensionless, like strain, and the constitutive equations do not have to include a density factor.

1.8 PREVIEW OF CONSTITUTIVE RELATIONS

The key concepts of Biot's 1941 poroelastic theory for an isotropic fluid-filled porous medium are contained in just two linear constitutive equations for the case of an isotropic applied stress field σ . In addition to σ , the other field quantities are the volumetric strain $\epsilon \equiv \delta V/V$, where V is the bulk volume, the increment of fluid content ζ , and the fluid pressure p . The volumetric strain $\delta V/V$ is taken to be positive in expansion and negative in contraction. Stress σ is positive if tensile and negative if compressive. Increment of fluid content ζ is positive for fluid added to the control volume and negative for fluid withdrawn from the control volume. Fluid pressure (pore pressure) p greater than atmospheric is positive. The constitutive equations express ϵ and ζ as a linear combination of σ and p :

$$\epsilon = a_{11}\sigma + a_{12}p \quad (1.6)$$

$$\zeta = a_{21}\sigma + a_{22}p \quad (1.7)$$

Generic coefficients a_{ij} are used in Eqns. 1.6 and 1.7 to emphasize the simple form of the constitutive equations. The first constitutive equation is a statement of the observation that changes in applied stress and pore pressure produce a fractional volume change. The second constitutive equation is a statement of the observation that changes in applied stress and pore pressure require fluid be added to or removed from storage.

Poroelastic constants are defined as ratios of field variables while maintaining various constraints on the elementary control volume. The physical meaning of each coefficient in Eqns. 1.6 and 1.7 is found by taking the ratio

of the change in a dependent variable relative to the change in an independent variable, while holding the remaining independent variable constant:

$$\begin{aligned}
 a_{11} &= \left. \frac{\delta \epsilon}{\delta \sigma} \right|_{p=0} \equiv \frac{1}{K} \\
 a_{12} &= \left. \frac{\delta \epsilon}{\delta p} \right|_{\sigma=0} \equiv \frac{1}{H} \\
 a_{21} &= \left. \frac{\delta \zeta}{\delta \sigma} \right|_{p=0} \equiv \frac{1}{H_1} \\
 a_{22} &= \left. \frac{\delta \zeta}{\delta p} \right|_{\sigma=0} \equiv \frac{1}{R}
 \end{aligned} \tag{1.8}$$

The coefficient $1/K$ is obtained by measuring the volumetric strain due to changes in applied stress *while holding pore pressure constant*.⁷ The state of constant pore pressure can be imagined to be enforced by inserting a tube into the rock and connecting it to a fluid reservoir at the same pressure. This state is called a *drained condition*, which is more general than the tube being vented to atmospheric pressure. Therefore, $1/K$ is the compressibility of the material measured under drained conditions, and K is the drained bulk modulus.

The coefficient $1/H$ is a property not encountered in ordinary elasticity. It describes how much the bulk volume changes due to a pore pressure change *while holding the applied stress constant*. By analogy with thermal expansion, it is called the *poroelastic expansion coefficient*. Energy considerations lead to the result that the coefficient $1/H$ is the same as $1/H_1$, that is, the linear transformation matrix is symmetric (see box).

The coefficient $1/R$ is a specific storage coefficient measured under conditions of constant applied stress; it is the ratio of the change in the volume of water added to storage per unit aquifer volume divided by the change in pore pressure. In this book the specific storage coefficient at constant stress is also called the *unconstrained* specific storage coefficient and is designated S_σ :

$$S_\sigma \equiv \frac{1}{R} \tag{1.9}$$

⁷ Constitutive equations are generally written in terms of absolute quantities rather than their differentials. Each quantity is considered to be relative to a reference state. Also, the expressions $p = 0$ and $\delta p = 0$ are considered equivalent, so that the words *constant pore pressure* or *drained conditions* are associated with both equations.

Potential Energy and Reciprocity. Biot (1941a) assumed the existence of a potential energy density:

$$U = \frac{1}{2}(\sigma\epsilon + p\zeta) \quad (1.13)$$

The potential energy density is required to satisfy the conditions

$$\frac{\partial U}{\partial \epsilon} = \sigma \quad \text{and} \quad \frac{\partial U}{\partial \zeta} = p \quad (1.14)$$

The mixed partial derivatives $\partial^2 U / \partial \zeta \partial \epsilon$ and $\partial^2 U / \partial \epsilon \partial \zeta$ must be equal; hence,

$$\frac{\partial \sigma}{\partial \zeta} = \frac{\partial p}{\partial \epsilon} \quad (1.15)$$

Solving Eqns. 1.6 and 1.7 for σ and p gives

$$\sigma = \frac{a_{22}}{\Delta}\epsilon - \frac{a_{12}}{\Delta}\zeta \quad (1.16)$$

$$p = -\frac{a_{21}}{\Delta}\epsilon + \frac{a_{11}}{\Delta}\zeta \quad (1.17)$$

where $\Delta = a_{11}a_{22} - a_{12}a_{21}$. Substituting Eqns. 1.16 and 1.17 into Eqn. 1.15 leads to the result that $a_{12} = a_{21}$, that is

$$\frac{1}{H} = \frac{1}{H_1} \quad (1.18)$$

Eqn. 1.18 implies that the following reciprocity relation holds: the volume of fluid expelled at constant fluid pressure due to an increase in compressive stress is the same as the unconstrained volume expansion due to an increase in pore pressure.

Biot also introduced the coefficient $1/M$, which is the specific storage coefficient *at constant strain*. It is called the *constrained*⁸ specific storage coefficient and designated S_ϵ . Micromechanical analysis (cf. Section 3.3) will show that the value of $1/R$ is determined by the compressibilities of the frame, the pores, the fluid, and the solid grains. Although $1/R$ has the same

⁸ The term *constrained* is used in soil mechanics (e.g., Lambe and Whitman, 1979) to mean uniaxially constrained. In this book *constrained* means three-dimensionally constrained, unless a qualifying adjective is used.

units as compressibility and can be expressed in terms of different compressibilities, its physical meaning is that of a storage coefficient.⁹

The introduction of three coefficients—drained compressibility ($1/K$), poroelastic expansion coefficient ($1/H$), and unconstrained specific storage coefficient ($1/R$)—completely characterizes the poroelastic response for isotropic applied stress. These three coefficients are the three independent components of a symmetric 2×2 matrix:

$$\begin{pmatrix} \frac{1}{K} & \frac{1}{H} \\ \frac{1}{H} & \frac{1}{R} \end{pmatrix} \quad (1.10)$$

The drained compressibility and the unconstrained storage coefficient are the diagonal components. The poroelastic expansion coefficient is the off-diagonal component. The symmetry condition means that $1/H$ has the same value for the coupling between strain and fluid pressure at constant stress as it does for the coupling between increment of fluid content and stress at constant pore pressure. Using Eqn. 1.8 in Eqns. 1.6 and 1.7 yields

$$\epsilon \equiv \frac{\delta V}{V} = \frac{1}{K}\sigma + \frac{1}{H}p \quad (1.11)$$

$$\zeta \equiv \frac{1}{H}\sigma + \frac{1}{R}p \quad (1.12)$$

Two additional coefficients—Skempton's coefficient B and constrained specific storage coefficient $S_\epsilon \equiv 1/M$ —are now introduced and expressed in terms of the three already defined. These examples illustrate further that poroelastic constants are ratios of field variables with specified constraints on the elementary control volume.

Skempton's coefficient is defined to be the ratio of the induced pore pressure to the change in applied stress for *undrained* conditions—that is, no fluid is allowed to move into or out of the control volume:

$$B \equiv - \left. \frac{\delta p}{\delta \sigma} \right|_{\zeta=0} \quad (1.19)$$

⁹ Bear (1972, p. 211) in his treatise on flow in porous media referred to Biot's theory as follows:

... assuming validity of Darcy's law and the generalized Hooke law [Eqns. 1.6 and 1.7], a theory was developed for flow in a consolidating medium, *without actually defining a storage coefficient* [emphasis added].

A negative sign is included in the definition because the sign convention for stress means that an increase in compressive stress, which induces a pore pressure increase, is a decrease in σ . The subscript equation $\zeta = 0$ is important because it expresses the undrained condition that no fluid is exchanged with the control volume. Substituting $\zeta = 0$ into Eqn. 1.12 shows that the induced pore pressure change is proportional to the applied stress: $\delta p = -(R/H)\delta\sigma$. Therefore,

$$B = \frac{R}{H} \quad (1.20)$$

If compressive stress is applied suddenly to a small volume of saturated porous material surrounded by an impermeable boundary, the induced pore pressure is B times the applied stress. Skempton's coefficient must lie between zero and one. Skempton's coefficient is a measure of how the applied stress is distributed between the skeletal framework and the fluid. It tends toward one for saturated soils, because the load is supported by the fluid. It tends toward zero for gas-filled pores because the load is supported by the framework.

The constrained specific storage coefficient, or specific storage coefficient at constant strain, is defined by

$$S_\epsilon \equiv \left. \frac{\delta\zeta}{\delta p} \right|_{\epsilon=0} \equiv \frac{1}{M} \quad (1.21)$$

where ϵ is the volumetric strain. The procedure for relating S_ϵ to the previously defined coefficients is to solve Eqn. 1.11 for σ and substitute it into Eqn. 1.12. The increment of fluid content is then expressed as a function of volumetric strain and pore pressure:

$$\zeta = \frac{K}{H}\epsilon + \left(\frac{1}{R} - \frac{K}{H^2} \right) p \quad (1.22)$$

Eqns. 1.21 and 1.22 show that

$$S_\epsilon = S_\sigma - \frac{K}{H^2} \quad (1.23)$$

Thus, the specific storage coefficient at constant strain is smaller than the specific storage coefficient at constant stress due to the constraint that the bulk volume remains constant. A schematic diagram comparing the constrained and unconstrained storage concepts is shown in Figure 1.6.

The ratio K/H is known as the *Biot-Willis coefficient* α :

$$\alpha \equiv \frac{K}{H} \quad (1.24)$$

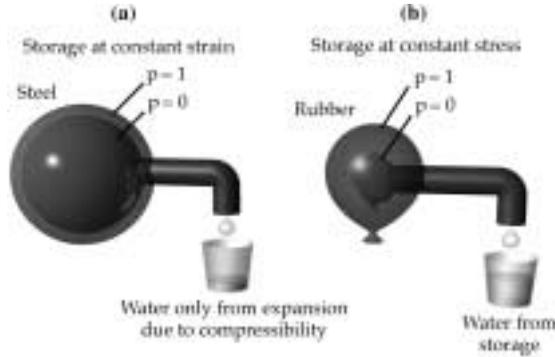


Figure 1.6: Constrained versus unconstrained storage. (a) Storage at constant strain is represented by the steel container. When the fluid pressure p decreases from one to zero, the only water released is from expansion of the water due to its compressibility, as the steel container does not change in volume. (b) Storage at constant stress is represented by the rubber balloon. When the fluid pressure p decreases from one to zero, the water released is due to both the expansion of the water because of its compressibility and the significant decrease in the volume of the balloon.

Eqn. 1.22 can then be rewritten as

$$\zeta = \alpha\epsilon + \frac{1}{M}p \quad (1.25)$$

Therefore, α is the ratio of volume of fluid that is added to storage divided by the change in bulk volume under the constraint that pore pressure remains constant.

In summary, three material constants—drained bulk modulus, poroelastic expansion coefficient, and unconstrained storage coefficient—characterize completely the linear, poroelastic response to volumetric deformation. Other constants, such as Skempton's coefficient and the constrained storage coefficient, can be derived from the three originally defined constants. A fourth independent constant, the shear modulus or the drained or undrained Poisson's ratio, is required for the complete poroelastic constitutive equations when shear stresses are present. Typical values of poroelastic constants for different rock types are given in Appendix C.

1.9 THERMOELASTIC ANALOGY

A complete mathematical analogy exists between poroelasticity and thermoelasticity (Biot, 1941a, 1956c; Rice and Cleary, 1976; Norris, 1992). Ther-

moelasticity combines the theory of heat conduction with constitutive equations that couple the stress and temperature fields. Temperature T corresponds to pore pressure, heat conduction corresponds to fluid flow, and entropy corresponds to fluid mass. In the thermoelastic equivalent of Eqn. 1.11, the volumetric strain is a function of mean stress and temperature:

$$\epsilon = \frac{\delta V}{V} = \frac{1}{K_t} \sigma + \alpha_t T \quad (1.26)$$

where $1/K_t$ is the isothermal compressibility and α_t is the volumetric thermal expansion coefficient [$^{\circ}K^{-1}$]. Identifying $1/K_t$ in Eqn. 1.26 as the isothermal ($T = 0$) compressibility is analogous to identifying $1/K$ in Eqn. 1.11 as the drained ($p = 0$) compressibility. Similarly, α_t is the stress-free thermal expansion coefficient, which corresponds to $1/H$, the stress-free poroelastic expansion coefficient.

The change in *specific* entropy δs (entropy per unit volume) for small changes from a reference equilibrium state at temperature T_o is given by the thermodynamic relationship

$$\delta s = \frac{h_Q}{T_o} \quad (1.27)$$

where h_Q is the quantity of heat added per unit volume. A change of entropy is proportional to heat transferred from storage. The analogue of δs in poroelasticity is the change of fluid mass content δm_f . The analogue of h_Q in poroelasticity is ζ , because ζ is a normalized volume of fluid added to or released from the porous material—just as h_Q is a normalized quantity of heat added to or released from the material. The variable m_f is a state variable of a poroelastic system, just as s is a state variable of a thermoelastic system. On the other hand, ζ and h_Q are not state variables because they are quantities that exist only when a change takes place within the system.

Biot (1956c) expressed the second thermoelastic constitutive equation as

$$\delta s = \alpha_t K_t \delta \epsilon + \frac{c_V}{T_o} T \quad (1.28)$$

where c_V is the specific heat at constant volume. Substituting Eqn. 1.27 into Eqn. 1.28 gives

$$h_Q = T_o \alpha_t K_t \epsilon + c_V T \quad (1.29)$$

Eqn. 1.29 can be compared with its poroelastic counterpart (Eqn. 1.25). The storage coefficient at constant strain, S_{ϵ} , is analogous to specific heat capacity

at constant volume, c_V . Isothermal conditions ($T = 0$) in thermoelasticity are analogous to drained conditions ($p = 0$) in poroelasticity. Isentropic conditions ($\delta s = 0$) and adiabatic conditions ($h_Q = 0$) are equivalent by Eqn. 1.27, and hence both isentropic and adiabatic conditions are equivalent to undrained conditions ($\delta m_f = \zeta = 0$) in poroelasticity.

To summarize, a one-to-one analogy has been established for the linear constitutive equations of poroelasticity and thermoelasticity. The thermal analogue to stress-free poroelastic strain induced by pore pressure changes (cf. Eqn. 1.11) is stress-free thermal strain induced by temperature changes. Thus, the volumetric thermal expansion corresponds to $1/H$ in Eqn. 1.11. The other coupling coefficient is the specific heat, which is the amount of heat required to change the temperature of the body, defined either for constant volume or constant entropy (adiabatic) conditions. Constant temperature (isothermal) and insulated (adiabatic) boundary conditions correspond to constant pore pressure (drained) and no-flow (undrained) boundary conditions, respectively, in the poroelastic case.

The completion of the analogy between poroelasticity and thermoelasticity requires a consideration of time-dependent transport. The mathematical form of Fourier's law for heat transport in response to temperature gradients is identical to Darcy's law for groundwater flow in response to pressure gradients. Heat conduction and groundwater flow are both thermodynamically irreversible processes. However, they are assumed to occur slowly enough that the system passes through a continuous sequence of equilibrium states.

The thermoelasticity literature (e.g., Boley and Weiner, 1985) is a valuable resource for poroelasticity because it can reinforce concepts of analogous poroelastic behavior and also allow some solutions to be transferred directly. For example, the opening paragraph of Nowacki's (1986) treatise on thermoelasticity is a good statement of the basic coupling between deformation and temperature:

A deformation of a body is inseparably connected with a change of its heat content and therefore with a change of the temperature distribution in the body. A deformation of a body which varies in time leads to temperature changes, and conversely. The internal energy of the body depends on both the temperature and the deformation. The science which deals with the investigation of the above coupled processes, is called thermoelasticity.

Biot's *increment of fluid content* is parallel to the term *change of heat content* used in thermoelasticity. In thermoelasticity, temperature changes produce thermal stresses, but stress changes do not significantly alter the temperature field in most materials. A similar uncoupling approximation exists in poroelasticity, in which the pore pressure field is calculated independently. Nowacki distinguishes between thermoelasticity and the theory of thermal

stresses as being the difference between full coupling and the assumption that the influence of deformation on the temperature field can be neglected. In poroelasticity it is also useful to make a similar distinction between full coupling and the assumption that changes in fluid pressure affect strain, but not vice versa.